Magnetic fields

Contents

MHD equations

Magnetic Field Lines

Magnetic fields in space

Dipole Fields

Dipole Field Lines

Interplanetary Magnetic Field

Superposition of Dipole and IMF Fields

Field from a Circular Loop

MHD Equations

Continuity

Conservation of matter requires total no. of particles N in a volume V can only change if there is a net flux of particles across the bounding surface S. Then

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t} dV = -\oint n\bar{u} \cdot d\bar{S} = -\int_V \nabla \cdot (n\bar{u}) dV$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\bar{u}) = 0$$

Motion

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} = -\nabla p + \bar{f}$$

Maxwell

$$\nabla \times \bar{B} = \bar{j}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \cdot \bar{E} = 0$$

$$\nabla \cdot \bar{B} = 0$$

Force

$$\bar{f} = \bar{j} \times \bar{B}$$

Ohm's Law

$$\bar{j} = \sigma\left(\bar{E} + \bar{v} \times \bar{B}\right)$$

where $\left(\bar{E} + \bar{v} \times \bar{B}\right)$ is the electric field felt by the observer (e.g. an electron)

Then (using $\nabla \times \left(\nabla \times \bar{B}\right) = -\nabla^2\bar{B} + \nabla\left(\nabla \cdot \bar{B}\right) = -\nabla^2\bar{B}$)

$$\frac{d\bar{v}}{dt} = -\nabla p + \frac{1}{\mu_0}\left(\nabla \times \bar{B}\right)\times \bar{B} = -\nabla\left[p + \frac{B^2}{2\mu_0}\right] + \frac{\left(\bar{B}\cdot\nabla\right)\bar{B}}{\mu_0}$$

$$\frac{\partial\bar{B}}{\partial t} = \nabla \times \left(\bar{v}\times\bar{B}\right) + \frac{1}{\mu_0}\nabla^2\bar{B}$$

Magnetic Field Lines

Consider a perfect conductor

$$\frac{\partial\bar{B}}{\partial t} = \nabla\times\left(\bar{v}\times\bar{B}\right)$$

Integrate over an arbitrary surface and use Stokes theorem:

$$\int_S \frac{\partial}{\partial t}\bar{B}\cdot d\bar{S} - \oint\left(\bar{v}\times\bar{B}\right)\cdot d\bar{l} = 0$$

$$\int \frac{\partial}{\partial t} + \oint \bar{B}\cdot\left(\bar{v}\times d\bar{l}\right) = \frac{d\Phi}{dt} = 0$$

The first term is the rate of change of flux through the fixed surface S, the second term is the additional increment caused by motion of the surface periphery moving with velocity v. i.e. conservation of flux.

The concept of a line of force or field line is an abstraction. In a perfect conductor the concept of field lines is useful. Consider a material line in the fluid, defined by intersecting surfaces. Choose these surfaces everywhere tangential to the magnetic field, at t = 0. Consequently the flux through the surfaces is zero and their intersection defines a field line at t = 0. The surfaces remain flux-free in the course of their motion. This is true for any two surfaces intersecting the same line at t = 0, so the line considered is always a field line. The field line is frozen into the fluid.

With finite conductivity, the surface integration gives

$$\frac{d\Phi}{dt} = \frac{1}{\mu_0}\int \nabla^2\bar{B}\cdot d\bar{S}$$

We find the same results for collisionless plasmas in the guiding center approximation as for a perfect conductor, as long as the electric drift is substituted for the fluid velocity. That is,

magnetic field lines are attached to guiding centers in magneto-plasmas in the zero order approximation (i.e. the MHD equations can be derived from the Boltzmann equation, with the guiding center approximation used. This is the approximation in which the cyclotron frequency is taken much larger than other frequencies).

The field line equation is

$$d\vec{l} \times \overline{B} = 0$$

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}$$

$$\frac{dl}{B} = const$$

Magnetic fields in space

a) Solar magnetic fields. Sunspots produce fields of 0.1 to 1 T, and are themselves produced by MHD dynamos(?). There is also a weak dipole field of 10^{-4}T. The dipole tilt is variable, with the dipole direction changing about every 11 years. The Inter Planetary Magnetic Field (IMF) is generated from the sun and stars, where the associated magnetic fields are convected outward with the solar and stellar winds.
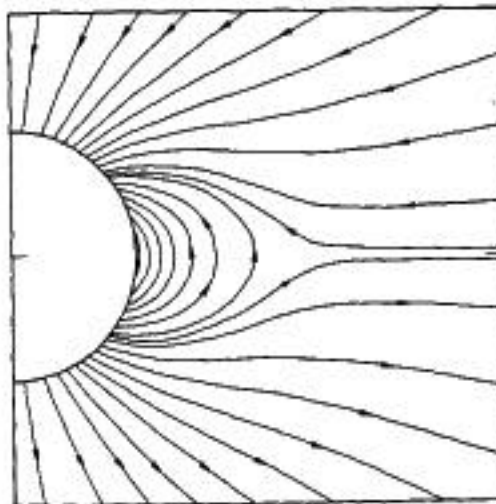

The sun's magnetic field lines, showing the general radial trend, with dipole structures and a field-reversed region.

b) Planet earth. Observations show an almost dipole like field; see below
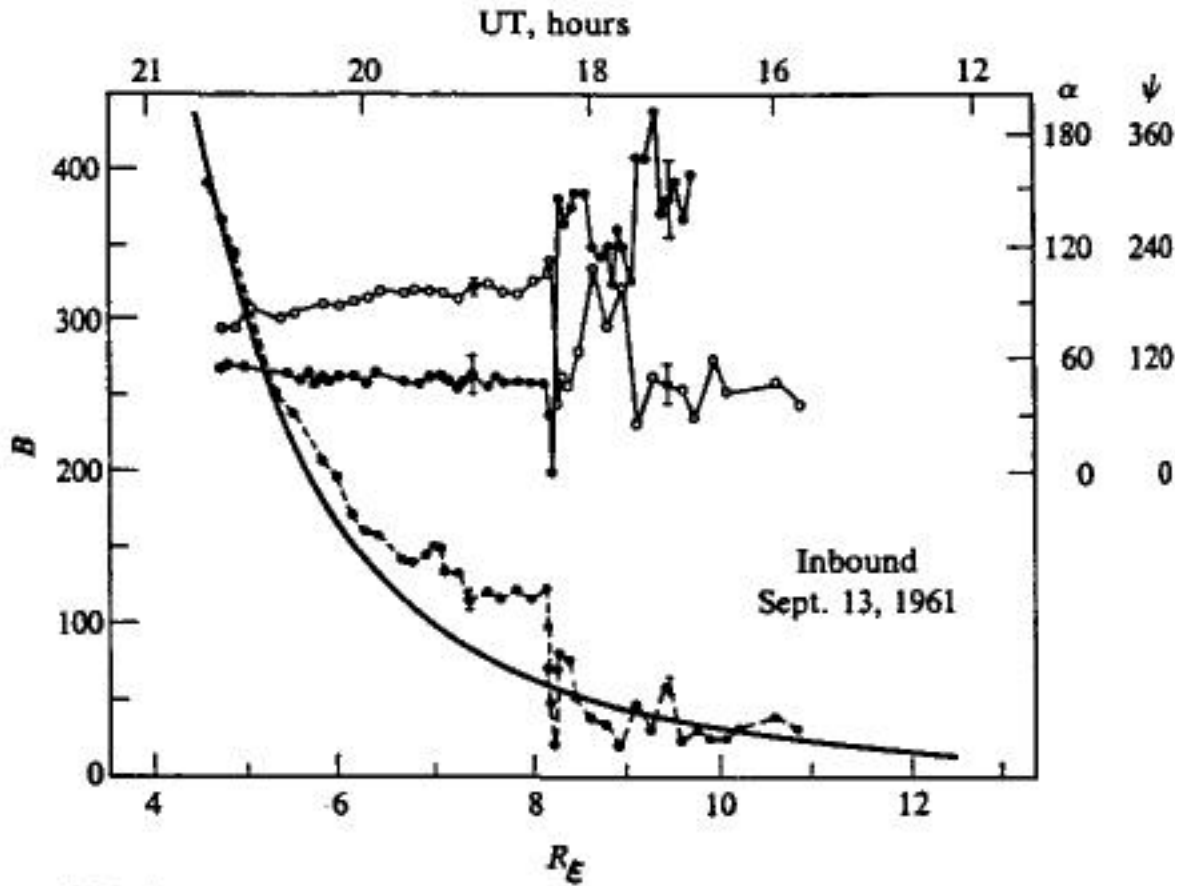
A figure of the magnetic field amplitude in nT from Explorer XII (1961). $R_E = 6375$ km. The solid line is the expected r^{-3} dependence of a dipole field. At 5 R_E the field diverges from that of a dipole. An abrupt change at 8 R_E is the crossing of the magnetopause boundary.
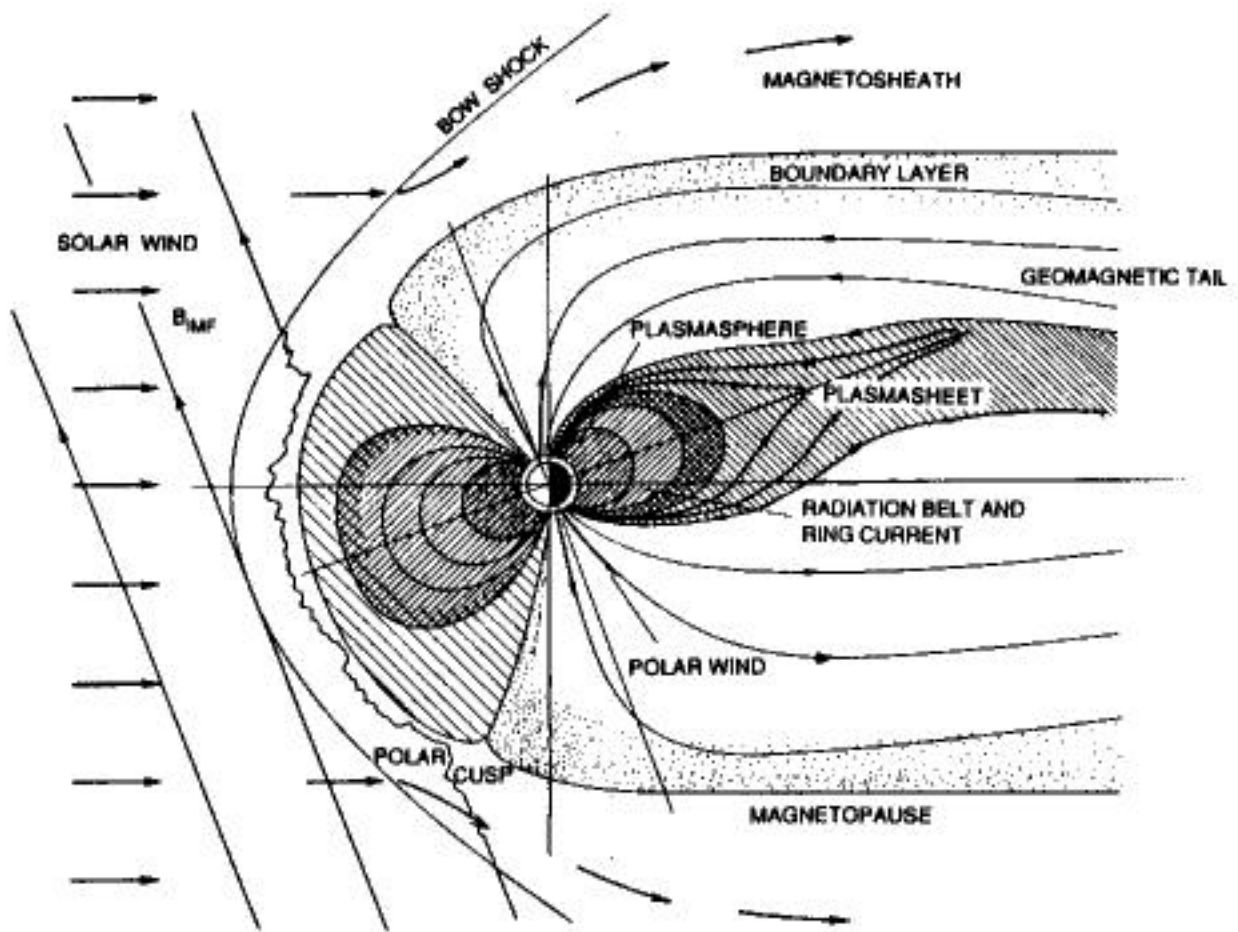
An illustration of the magnetosphere in the noon - midnight plane.

The magnetopause is the separation between a planets B fields and the solar wind. The inner boundary of the ionosphere separates the conducting ionosphere from the neutral atmosphere.

Dipole Fields

The dipole magnetic field is represented by a magnetic moment $\overline{M}$. Work in a spherical coordinate system as below.
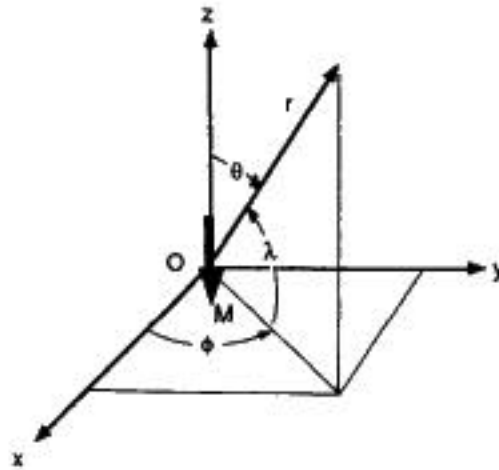
Geometry of the earth's dipole field

Outside the planet $\nabla \times \bar{B} = 0$. The field B is obtained either from a scalar or a vector potential. For the scalar potential

$$\bar{B} = -\nabla\psi$$

i.e. $\nabla \cdot \bar{B} = -\nabla \cdot (\nabla\psi) = 0$, and ψ obeys Laplace's equation. Surfaces of constant ψ are called equipotential surfaces, and these are orthogonal to the magnetic field surfaces. Then we write

$$\psi = -\frac{\mu_0}{4\pi}\bar{M} \cdot \nabla\frac{1}{r} = -\frac{\mu_0 M \cos(\theta)}{4\pi \; r^2} = -\frac{\mu_0 M \sin(\lambda)}{4\pi \; r^2}$$

Note $\bar{M} = -M\hat{z}$, $\lambda = \frac{\pi}{2} - \theta$ with $\hat{z}$ a unit vector, θ the colatitude and λ the latitude. Then the gradient gives

$$B_r = -\frac{\partial\psi}{\partial r} = -\frac{\mu_0 M}{2\pi}\frac{\sin(\lambda)}{r^3}$$

$$B_\lambda = -\frac{1}{r}\frac{\partial\psi}{\partial\lambda} = \frac{\mu_0 M}{4\pi}\frac{\cos(\lambda)}{r^3}$$

$$B_\phi = -\frac{1}{r\cos(\lambda)}\frac{\partial\psi}{\partial\phi} = 0$$

The field intensity $|\bar{B}| = \sqrt{B_r^2 + B_\lambda^2 + B_\phi^2}$, so that the field magnitude B at a distance r and latitude λ is given by

$$B(r, \theta) = \frac{\mu_0 M}{4\pi r^3}\left(1 + 3\sin^2(\theta)^{1/2}\right)$$

The field intensity is smallest at the equator and largest at the poles

Note we could equally well have utilized a vector potential, defined by

$$\overline{B} = \nabla \times \overline{A}$$

Then

$$\nabla \times \overline{B} = \nabla \times \left(\nabla \times \overline{A}\right) = \nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2\overline{A} = -\nabla^2\overline{A}$$

where the choice of a 'coulomb gauge' $\nabla \cdot \overline{A} = 0$ is used. Then

$$\nabla^2\overline{A} = -\mu_0\overline{j}$$

As discussed before, $\overline{B}$ is tangent to $d\overline{l}$ along which the magnetic field line is defined.

Dipole Field Lines

The dipole filed lines are given by the solution of

$$\frac{dl}{B} = const$$

i.e.

$$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{r\sin(\theta)d\phi}{B_\phi}$$

now use $d\phi = -d\theta$, and $B_\phi = -B_\theta$, to get

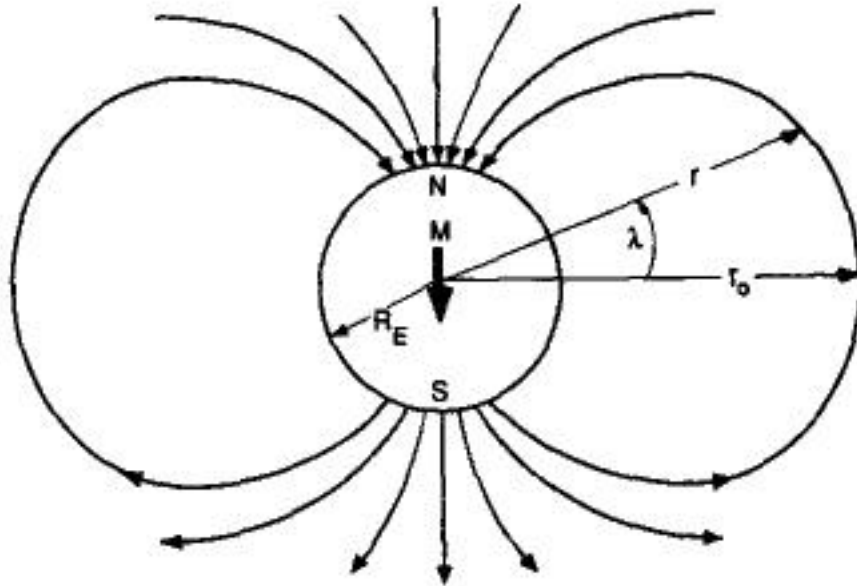$$d\phi = 0; \quad \frac{dr}{rd\theta} = \frac{B_r}{B_\theta} = -\frac{2\sin(\theta)}{\cos(\theta)}$$

the second equation can be written as

$$\frac{dr}{r} = \frac{2d(\cos(\theta))}{\cos(\theta)}$$

so that integration yields the dipole field line equation

$$\phi = \phi_0; \quad r = r_0\cos^2(\theta)$$

with r_0 the equatorial crossing distance at $\quad = 0$

Locus of a dipole line of force

## Interplanetary Magnetic Field

B outside the magnetosphere. For the earth, but near the earth, the IMF varies from a quiet few nT to 20 nT during solar flares. The IMF is dominantly in the ecliptic plane, but there is a significant amplitude perpendicular to this

$$\overline{B}_{IMF} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

B_x, B_y are in the ecliptic plane, B_z is perpendicular to it. $\hat{x}$ points at the sun, $\hat{z}$ is perpendicular to the ecliptic.

Superposition of Dipole and IMF Fields

i.e. $\overline{B}_T = \overline{B}_{IMF} + \overline{B}_D$
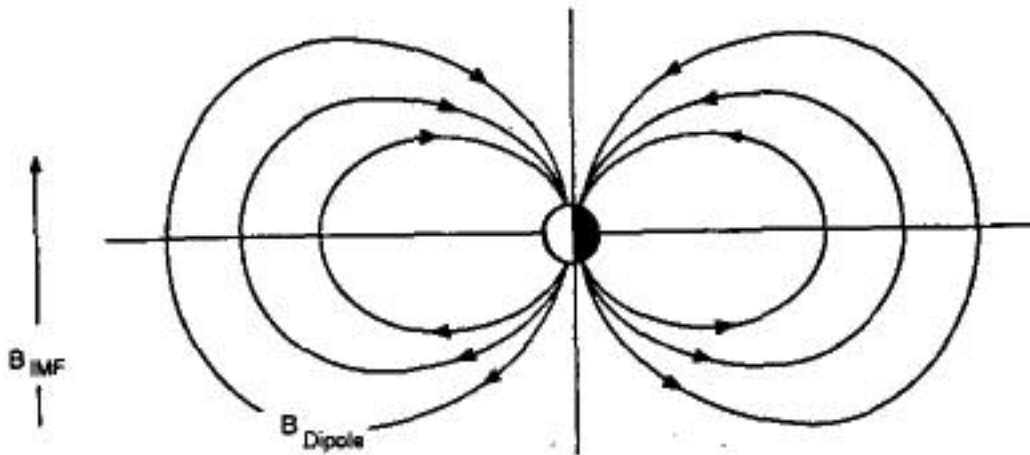
Superposition of North IMF and Dipole field

North IMF.

For a northward IMF, $\overline{B}_{IMF} = B_0 \hat{z}$. The magnetic potential in the reference frame of planet earth is

$$\psi_{IMF} = -B_0 \hat{z} = -B_0 r \sin(\theta)$$

so that

$$B_{IMF}(r) = -\frac{\partial \psi}{\partial r} = B_0 \sin(\theta)$$

$$B_{IMF}(\theta) = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = B_0 \cos(\theta)$$

Added to the earth's dipole fields we have

$$B_T(r) = -\frac{\mu_0 M}{2}\frac{\sin(\theta)}{r^3} + B_0 \sin(\theta)$$

$$B_T(\theta) = \frac{\mu_0 M}{4}\frac{\cos(\theta)}{r^3} + B_0 \cos(\theta)$$

The total field intensity is given by

$$B_T(r,\theta) = \left[\frac{\mu_0^2 M^2}{16\pi^2 r^6}\left(1 + 3\sin^2(\theta)\right) + B_0^2 + \frac{\mu_0 M B_0}{2\pi r^3}\left(1 - 3\sin^2(\theta)\right) \right]^{\frac{1}{2}}$$

Now to get the field line equation, use $\dfrac{dr}{B_r} = \dfrac{rd\phi}{B_\phi} = \dfrac{r\sin(\phi)d\lambda}{B_\lambda}$ and obtain $d\phi = 0$, and

$$\frac{1 + \dfrac{4\pi B_0 r^3}{\mu_0 M}}{1 + \dfrac{2\pi B_0 r^3}{\mu_0 M}} \frac{dr}{r} = -\frac{2\sin(\lambda)}{\cos(\lambda)}d\lambda$$

Integration yields (check by differentiating) $\phi = \phi_0$ and

$$\frac{r}{1 - \dfrac{4\pi B_0 r^3}{2\mu_0 M}} = \frac{r_0}{1 - \dfrac{4\pi B_0 r_0^3}{2\mu_0 M}}\cos^2(\lambda)$$

where as before r_0 is the equatorial crossing distance at $\lambda = 0$
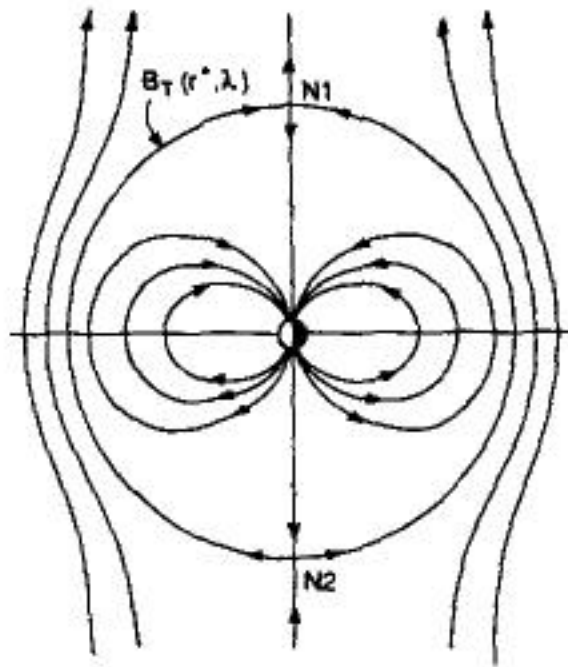
Magnetic neutral points

Let $B_T(r) = 0$ which happens at

$$r_n = \left[\frac{\mu_0 M}{2\pi B_0}\right]^{-\frac{1}{3}}$$

Set $B_T(\ ) = 0$; this can only happen at $\quad = \pm\dfrac{}{2}$. Therefore with the north IMF there are two

neutral points located at r_n and $\quad = \pm\dfrac{}{2}$. The surface of the sphere of radius r_n has a field

$$B_T(r_n,\ ) = \frac{3}{2} B_0 \left(1 - \sin^2(\ )\right)^{1/2}$$

South IMF.

For a southward IMF, $\overline{B}_{IMF} = -B_0 \hat{z}$, and

$$B_T(r) = -\frac{\mu_0 M}{2} \frac{\sin(\ )}{r^3} - B_0 \sin(\ )$$

$$B_T(\ ) = \frac{\mu_0 M}{4} \frac{\cos(\ )}{r^3} - B_0 \cos(\ )$$

The total field intensity is given by

$$B_T(r,\ ) = \left[\frac{\mu_0^2 M^2}{16\ ^2 r^6}\left(1 + 3\sin^2(\ )\right) + B_0^2 - \frac{\mu_0 M B_0}{2\ r^3}\left(1 - 3\sin^2(\ )\right) \right]^{\frac{1}{2}}$$

The equation for the total field (field line equation) is $\quad = \quad_0$ and

$$\frac{r}{1 + \dfrac{4\ B_0 r^3}{2\mu_0 M}} = \frac{r_0}{1 + \dfrac{4\ B_0 r_0^3}{2\mu_0 M}} \cos^2(\ )$$

where as before r_0 is the equatorial crossing distance at $\quad = 0$

neutral lines

The total field $= 0$ when $B_T(\ ) = 0$, i.e. when $\quad r_n = \left(\dfrac{\mu_0 M}{4\ B_0}\right)^{\frac{1}{3}}$.

This must occur ($B_T(r) = 0$) when $\quad = 0$., i.e. on the equator. The distance to the neutral line is not the same as to the neutral point. There are 3 kinds of field lines. Type 1 is solar, type 2 is anchored to the earth, type three is anchored both to the sun and the earth. That is, high latitudes are open to space. That this may actually happen is suggested by the observation of the direct entry of solar protons following a solar flare event. These protons cause blackouts for radio

wave propagation. The model predicts that the 'polar cap' is reduced to a point if the IMF is north. This is also consistent with observations.
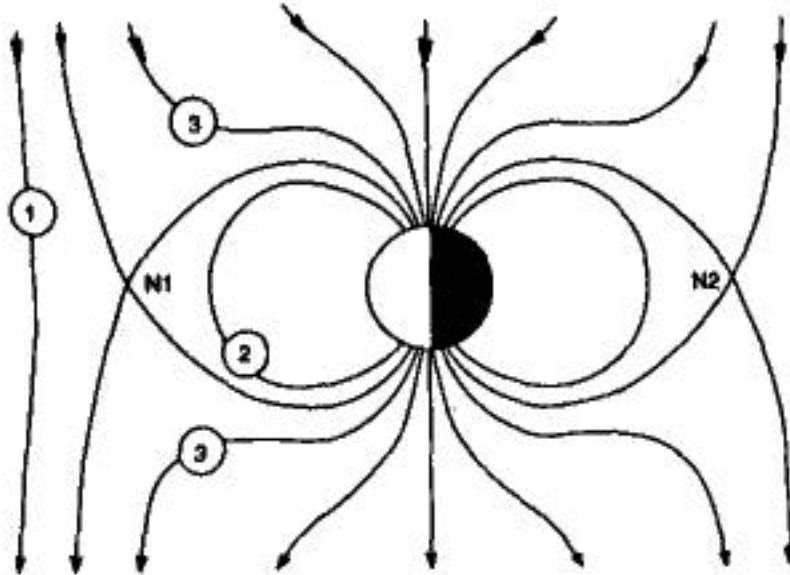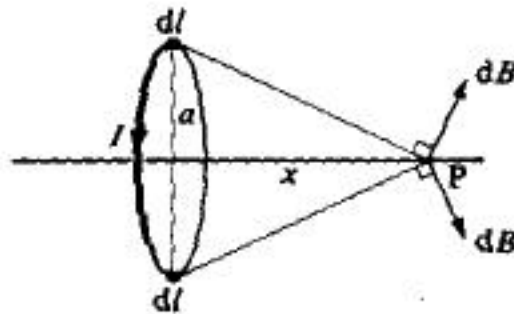

Field from a Circular Loop

$$d\bar{B} = \frac{\mu_0 I d\bar{l} \times \bar{r}}{4 \; r^3}$$
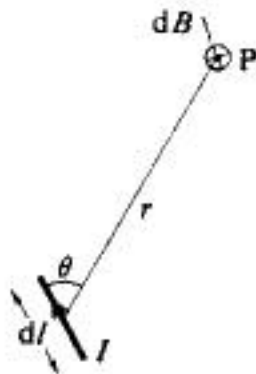
A current element and its relation to the point P.

Derivation of flux density due to a circular coil using the current element formula.

Shown are flux densities due to two elements at opposite ends of a diameter; each is equal to

$$dB = \frac{\mu_0 I dl}{4 \ \left(x^2 + r^2\right)}$$

The components perpendicular to the axis cancel, while the components along the axis add. This applies to each pair of elements, so the total B at P is the sum of all resolved parts along x. At the origin (center of the coil) r = a and x = 0. Now dl = a d , so

$$B = \frac{2\mu_0 I a d}{4 \ a^2} = \frac{\mu_0 I}{2a}$$